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Modeling Returns of Stock Indexes through Fractional Brownian Motion Combined with Jump Processes and Modulated by Markov Chains

Modelado de rendimientos de índices bursátiles mediante movimiento fraccional browniano combinado con procesos de saltos y modulado por cadenas de Markov

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Abstract

This paper's aim is to extend the Durham and Park's (2012) model by incorporating the market fractional behavior. The extension examines the stochastic dynamics of stock indexes for several of the world´s main economies (US, Eurozone, UK and Japan), as well as emerging markets (China, Brazil and Mexico) during 1994-2017. The proposed model assumes that the returns are driven by fractional Brownian motions combined with Poisson processes and modulated by Markov chains. Risk factors such as: idiosyncratic volatility, market volatility, volatility of volatility were incorporated. To accomplish the purpose of the extension, Jump-GARCH and Markov regime-switching models were estimated, the Hurst coefficient was calculated and jumps behaviour was analysed during crisis periods. It was considered that the model accurately describes the stochastic dynamics of the stock indexes returns. The main empirical findings are that the USA stock market remains in high volatility most of the time, that the Brazil

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Stock Market has the highest intensity of jumps, and that the mean return is the highest for the IPC and the lowest for Nikkei Index during the period under study.

JEL Classification: N20, B23, C02.

***Keywords**: Stock Index Return, Fractional Brownian Motion, Markov Regime-switching, Jump Processes.*

Resumen

Esta investigación tiene como propósito extender el modelo de Durham y Park (2012) al incorporar la conducta fraccional del mercado. La extensión examina la dinámica estocástica de los índices bursátiles de economías desarrolladas como en EUA, Eurozona, Reino Unido y Japón, así como los mercados emergentes de China, Brasil y México durante 1994-2017. Nuestro modelo supone que los rendimientos son conducidos por movimientos brownianos fraccionales, combinados con procesos de Poisson y modulados por cadenas de Markov. Se incorporaron factores de riesgo como: volatilidad idiosincrática, volatilidad del mercado y volatilidad de la volatilidad. Para ello, se estimaron los modelos Jump-GARCH y de cambio de régimen Markoviano, se calcularon los coeficientes de Hurst y se analizó el comportamiento de los saltos en periodos de crisis. Se encontró que el modelo propuesto describe adecuadamente la dinámica estocástica de los rendimientos de los índices bursátiles estudiados. Los principales hallazgos empíricos son que el mercado de valores de EUA, se mantiene en alta volatilidad la mayor parte del tiempo, que el mercado de valores de Brasil tiene la mayor intensidad de saltos, que el rendimiento medio más alto lo presenta el IPC y el más bajo el índice Nikkei durante el período de estudio.

Clasificación JEL: N20, B23, C02.

Palabras clave: rendimiento de índices de acciones, movimiento fraccional browniano, cambio de régimen markoviano, procesos de saltos.

Introduction

Over a long period of time, several researchers have dealt with the stochastic dynamics and return probability distributions of the stock markets; however, there are still irregularities and stylized facts that need to be explained. The development of the stock market indexes has followed complex dynamics derived from intricate global investment strategies, requiring more sophisticated models and tools. Most of the models in the specialized literature can be broadly classified into two large groups: Models seeking to explain the fundamental value of stock and models describing stock prices dynamics (Krause, 2001). For the latter, research has been focused on the volatility of aggregate stock markets through cross-section analysis (Ang, 2004). Some other studies have used stochastic calculus to model stock returns and time-varying volatility, for example, Christoffersen et al. (2009) built a two-factor stochastic volatility model useful to generate time-varying correlation. Along the same line, Johnson (2002) developed a stochastic volatility model with time-varying correlation between returns and volatility, An et al. (2014) used option volatilities cross-section analysis to forecast stock returns, and López-Herrera et al. (2009) studied the long-term dependence on returns and volatilities.

Several studies focused on return distributions with time-varying moments, Carr and Wu (2007) proposed a stochastic skew model for foreign exchange rates; Pham and Touzi (1996) explored the stochastic volatility on equilibrium state prices; Durham and Park (2012) focussed on stochastic volatility in stock returns and found that return distributions have time-varying skewness and kurtosis. Young *et al*. (2013) encountered that stocks with high sensitivity to innovation in implied market volatility and skewness exhibit, on average, low returns; finally, Harvey and Siddique (1999) examined time-varying skewness through a GARCH model, and suggested that the relation between stock returns skewness and variance are linked to the seasonal variations in the conditional moments

Another factor that has been relevant when examining returns dynamics is the volatility of volatility.[1] Some studies have found evidence that the variance risk premium depends on the volatility of volatility. For instance, Das and Sundaram (1999) examined the volatility of volatility and the correlation between the innovations in asset pricing. Also, Durham and Park (2012) developed a mixed jump-diffusion process on options with volatility of volatility (*cf.* Ang *et al*., 2006).

An important characteristic of the stock markets is the presence of unexpected and sudden jumps. Cremers *et al*. (2015) suggested, by using cross-section analysis, that stock returns have high sensitivity to jumps and volatility risks. Moreover, Du and Kapadia (2011) argued that the index VIX (Chicago Board Options Exchange Market Volatility Index) has a critical degree of bias related to jumps. Also, Branger *et al*. (2007) proposed

[1] Volatility of volatility is a measure of volatility expected of the n-day forward price of the volatility and this drives nearby volatility options price.

an equilibrium model with jumps and stochastic volatility to describe the dynamics of stock returns. Along the same line, Heston (1993) dealt with Poisson jumps through a stochastic volatility model with returns affected by three factors: diffusive volatility shocks, diffusive price shocks, and price jumps. Finally, Bates (2008) examined investors' behaviour towards crash risk and extreme and common events related to stock returns jumps. The author considers jumps and diffusion risks separately (*cf*. Johnson, 2002).

There are other studies addressing options and futures markets, such as Durham and Park *et al*. (2012). These authors developed a stochastic volatility model to assess several characteristics that are consistent with variation in the shape of return distributions by including regime-switching to feature random changes in the volatility of volatility, leverage effect, and jump intensity. Santa-Clara and Yan (2010) presented a model of option prices when the volatility of the diffusion shocks and the intensity of the jumps change over time, and show that diffusive volatility and jump intensity capture the ex-ante risk assessed by investors of the S&P500 index options. Moreover, Vallejo and Venegas-Martínez (2017) modeled the dynamics of asset prices with time-inhomogeneous Markov chains and applying fractional Brownian motion with multiple Poisson jumps (*cf*. Venegas-Martínez, 2001 and 2008).

The above investigations have highlighted the importance of including the effect of volatility, volatility of volatility, unexpected jumps, and regime-switching on stock returns. The hypothesis of this paper establishes that the returns of stock indexes are properly driven by fractional Brownian motion implying long-term memory. This article mainly extends current studies from Durham and Park's (2012) by modelling fractional behaviour of the stock markets and by examining the performance of returns and their jumps to describe the stochastic dynamics of stock indexes of various economies (US, Eurozone, UK and Japan, China, Brazil, and México) during 1994-2017. To accomplish the purpose of the extension, Jump-GARCH and Markov regime-switching models were estimated, and the Hurst coefficients were calculated using different econometric programs (E-views, RATS and Pracma) and R software.

This paper is organized as follows: the first section presents the extended stochastic model of stock index returns; section two describes the data and defines the endogenous and exogenous variables; section three calibrates the proposed model; and finally the conclusions are provided.

1. Modelling Stock Index Returns

This section presents the theoretical background needed to model the dynamics of stock indexes returns by using fractional Brownian motion combined with Poisson process modulated with Markov switching-regime stochastic volatility. Most of the empirical studies suggest that market volatility varies over time and stocks with high sensitivity to both jump and volatility risks have low expected returns (Cremers *et al*. 2015). Durham and Park (2012) proposed a Markov regime-switching model of both volatility of volatility and jump intensity to determine the skewness and kurtosis of stock returns. Also, Vallejo-Jiménez and Venegas-Martínez (2017) developed a model that explains the dynamics of asset prices that are driven by multiple jumps, fractional Brownian motion, and Markov regime switching.[2]

In the proposed multifactor risk model, stock returns are driven by the fractional Brownian motion, $\mathrm{d}B_{1t}^{H}$ and $\mathrm{d}B_{2t}^{H}$, combined with Poisson jumps, $\mathrm{d}N_t$, and modulated by Markov regime switching, $E=\{\sigma_1,\sigma_2\}$, aligned with Durham and Park (2012):

$$\mathrm{d}y_t=\left(\mu-\tfrac{1}{2}v_t^2\right)\mathrm{d}t+v_t\mathrm{d}B_{1t}^{H} \tag{1}$$

$$\mathrm{d}v_t=a(b-v_t)\mathrm{d}t+\sigma_i\mathrm{d}B_{2t}^{H}+\gamma\mathrm{d}N_t \tag{2}$$

where $\mathrm{d}y_t$ is a dependent variable determining the dynamics of the stock index return, $\mathrm{d}B_{1t}^{H}$ and $\mathrm{d}Bt_{2t}^{H}$ are independent fractional Brownian motions, H is the Hurst parameter, μ is the annual mean (trend parameter) of returns, $\mathrm{d}N_t$ is a Poisson jumps, v_t is the idiosyncratic volatility, E is the regime state (low volatility and high volatility), σ_i is the volatility state, dv_t is the volatility of volatility, a is the speed adjustment parameter, b is the long run mean (mean revering), and γ is the mean jump size.[3]

A Markov regime-switching process (Hamilton, 2005) is a nonlinear time series model that integrates multiple structures to explain the be-

2 See also Christoffersen et al. (2009) and Ang et al. (2006).

3 Other papers dealing with jump-diffusion processes are Venegas-Martínez (2000) and (2001) and Venegas-Martínez and González-Aréchiga (2000).

haviour of a state variable in different regimes. The probabilities of switching from state to state are given by the following transition matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

The fractional Brownian motion B_t^H is defined on a fixed probability space with its augmented filtration $(\Omega, F, (F_t)_{t\in[0,T]}, P)$ and $H \in (0,1)$ is the Hurst coefficient (Taqqu, 2013). A Hurst coefficient H larger than 0.5 measures long-term memory of time series. It describes the irregularity of the motion, predict the stock return and reflect the autocorrelation on returns. It is worth mentioning that if $H \neq \frac{1}{2}$, then B_t^H is not a semimartingale (Mandelbrot and Van Ness, 1968), and when H is smaller that 0.5, it reflects a mean reverting effect. This can be summarized as:[4]

H = ½, the process is Brownian motion or Wiener process.

H > ½, the increments are positively correlated (long memory).

H < ½, the increments are negatively correlated (mean reverting).

It is important to point out that Cajueiro and Tabak (2005) find that the Hurst coefficient on Brazilian stock market is time-varying; Jamdee and Los (2005) showed that European options have long memory and are dependent on volatility; and Bender (2000) suggested that the law of one price holds in a market where the stock is driven by fractional Brownian motion.

The ARCH model is briefly review, which is useful to explain the large residuals' trend to cluster together (Engle, 1982). The ARCH model is given by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \, \varepsilon_{t-i}^2 \qquad (3)$$

where σ_t^2 is the conditional variance, ω and α are unknown parameters, and ε_{t-i}^2 is the lag of the random error term. In the GARCH model the

[4] Duncan and Pasik-Duncan (1991) introduced an integration theory named Wick-Itô-Skorohod integral for fractional Brownian motion.

variance term depends on the lagged variance as well as the lagged square residuals. This model allows to evaluate different types of persistence in volatility (Bollerslev, 1986). The GARCH model is represented by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \, \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i \, \sigma_{t-j}^2 \qquad \omega, \alpha, \beta > 0 \tag{4}$$

where σ_t^2 is the conditional variance, ω and α and β are unknown parameters, ε_{t-i}^2 is the lag of the random deviation term, σ_{t-j}^2 is the lag of the variance, α is the component of the influence of random deviation in the previous period, β is the component of the variance in the previous period, and $\alpha + \beta$ is the level of persistence. Extending these approaches, the Jump-GARCH model is an alternative for modelling the dynamics of stock indexes when sudden and unexpected jumps occur (Chen, Lin and Lin, 2013). In this case, two stochastic innovations, $\varepsilon_{t,1}$ and $\varepsilon_{t,2}$, capture the dynamic of the return with no jump and jump, respectively. The innovations $\varepsilon_{t,1}$ and $\varepsilon_{t,2}$ are independent and satisfy:

$$\varepsilon_t = \varepsilon_{t,1} + \varepsilon_{t,2} \tag{5}$$

The first innovation refers to market stability with no jumps, thus:

$$\varepsilon_{t,1} = \sigma_t u_t, \qquad u_t \sim N(0,1) \tag{6}$$

and

$$\mathrm{E}\left(\varepsilon_{t,1} | \, y_{t-1} \right) = 0. \tag{7}$$

The second innovation describes an unexpected jump when an unusual event occurs. The returns of the stock market are impacted by an unexpected event. The distribution of jumps follows a Poisson distribution and λ is the parameter of the jump intensity, hence:

$$\varepsilon_{t,2} = N_t - \mathrm{E}(N_t | y_{t-1}) = \sum_{k=1}^{n_t} \gamma_{t,k} - \theta \lambda_t, \qquad n_t | y_{t-1} \sim P(\lambda_t) \tag{8}$$

and

$$\varepsilon_{t,2} = \mathrm{E}\left(\varepsilon_{t,2} | \, y_{t-1} \right) = 0 \tag{9}$$

where y_t stands for the dynamics of the return, N_t is the jump component, $\gamma_{t,k}$ is the jump size, λ_t is the jump intensity, θ is the component of jump intensity, and n_t denotes the number of jumps. The Poisson process N_t with intensity parameter λ satisfies:

$$\mathrm{P}\{\text{One jump on d}t\} = \mathrm{P}\{\mathrm{d}N_t = 1\} = \lambda dt + o(\mathrm{d}t) \tag{10}$$

$$\mathrm{P}\{\text{None jump on } dt\} = \mathrm{P}\{\mathrm{d}N_t = 0\} = 1 - \lambda \mathrm{d}t + o(\mathrm{d}t). \tag{11}$$

Hence,

$$\mathrm{P}\{\text{More that one jump on d}t\} = \mathrm{P}\{\mathrm{d}N_t \geq 1\} = o(\mathrm{d}t). \tag{12}$$

Then,

$$\mathrm{E}[\mathrm{d}N_t] = \mathrm{Var}[\mathrm{d}N_t] = \lambda \mathrm{d}t, \tag{13}$$

$$\mathrm{Cov}(\sigma_t u_t, N_t) = 0 \tag{14}$$

2. Data description

This section, aims to find out how well the proposed model captures and describes the dynamics of the stock index returns under study. The data for the US (S&P 500), Eurozone (EuroStoxx50), United of Kingdom (FTSE100), Japan (Nikkei), China (Hang Seng), México (IPC) and Brazil (Bovespa) were obtained from Bloomberg and includes daily returns of each stock index. The USA is considered as a benchmark since it is the world´s largest economy and it has the biggest financial market.[5] Hong Kong is the most important

[5] Investors of financial markets often take decisions based on Eurozone and UK economic data.

financial center in Asia; Japan is a highly developed economy in Asia and it has the largest electronic goods industry; Mexico and Brazil exhibit the best macroeconomic indicators of Latin America.

The analysis sample period begins in January 1994 and ends in December 2017 (5980 daily returns for each stock index). The purpose of this study is to capture in the extension proposed the dynamics of stock market indexes before, during and after crisis periods. The most relevant extreme events are the Asian financial crisis, the bubble dot com in 2001, the subprime mortgage recession in 2008, the Eurozone debt crisis in 2011, the Brexit in June 2016, and the power takeover of president Trump in December 2016. The idiosyncratic volatility is represented by the standard deviation. The market volatility is calculated through the VIX index, which is a measure of the expected volatility of the US stock market during 30 days, calculated from real-time mid quote prices of S&P500 call and put options index (CBOE). Finally, the volatility of volatility is the square return of measure by the VIX index. The parameters for Markov regime switching were estimated using *E-views* software, Jump-GARCH with Rats software and the Hurst coefficient was calculated using the Pracma package programmed in R.

3. Empirical analysis

The results of the Markov regime-switching models describing the degree of volatility of the previous period of the returns of S&P500, Eurostoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa indexes are shown in Table 1.

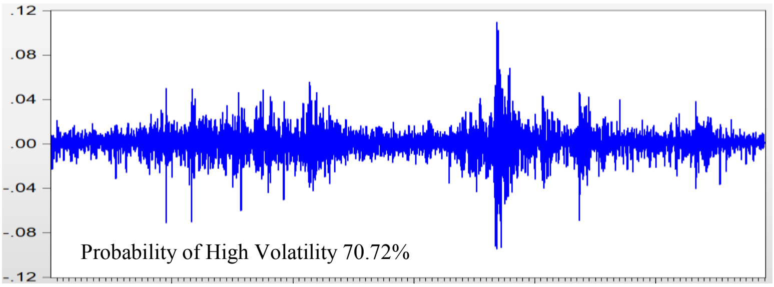
Table 1. Transition probabilities of the Markov Regime-Switching model

Probabilities	S&P500	EuroStoxx50	FTSE100	IPC	Bovespa	Nikkei	Hang Seng
p11	0.41424	0.52744	0.54349	0.58354	0.56479	0.46037	0.48052
p12	0.58576	0.47255	0.45650	0.41646	0.43521	0.53963	0.51949
p21	0.29279	0.40675	0.50486	0.38694	0.44869	0.54721	0.37474
p22	0.70721	0.59324	0.49513	0.61306	0.55131	0.45279	0.62526

Source: Prepared by authors with Bloomberg data and E-views software.

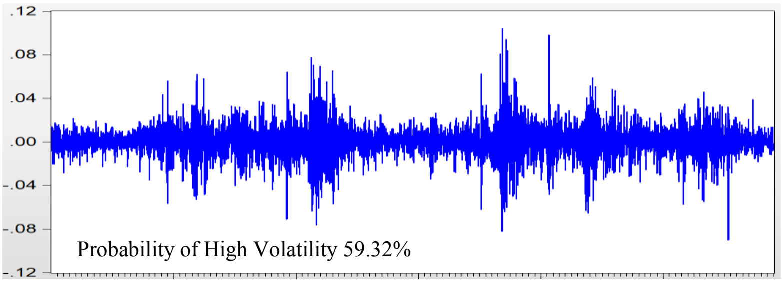
Table 1 shows that the S&P500 has 70% of probability to stay in high volatility from one period to another, followed by the Hang Seng with 62%; while, IPC, Bovespa and FTSE100 have a bigger probability to stay in low volatility than the others. Indexes FTSE100 and Nikkei are more probable to change from high to low volatility. The S&P500 has 58% of probability to transit to high volatility, and just a 29% of probability that this index will changed from high to low volatility. Figures 1-7 show the returns of S&P500, Eurostoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa indexes from 1994 to 2017 (the *x*-axis measures the number of days) these have higher jumps on the most relevant economic event as bubbles, crises and politiques decisions around the world; see Figures 1 to 7.

Figure 1. Returns of S&P 500 (1994-2017)



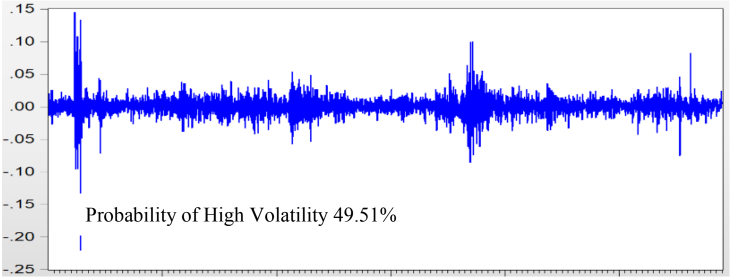
Source: Prepared by authors with Bloomberg data and E-Views.

Figure 2. Returns of EuroStoxx50 (1994-2017)



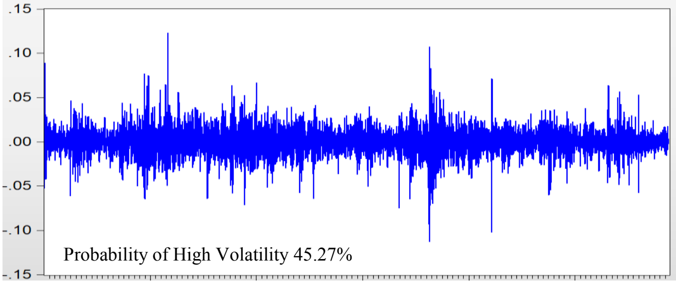
Source: Prepared by authors with Bloomberg data and E-Views.

Figure 3. Returns of FTSE100 (1994-2017)



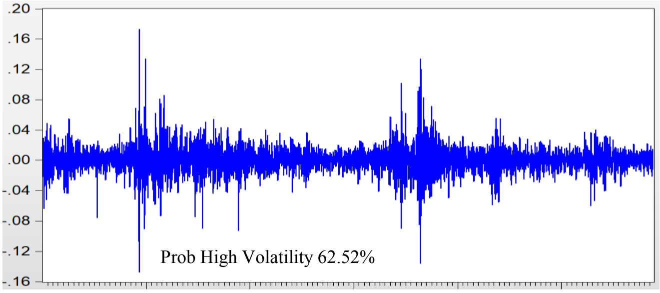
Source: Prepared by authors with Bloomberg data and E-Views.

Figure 4. Returns of Nikkei (1994-2017)



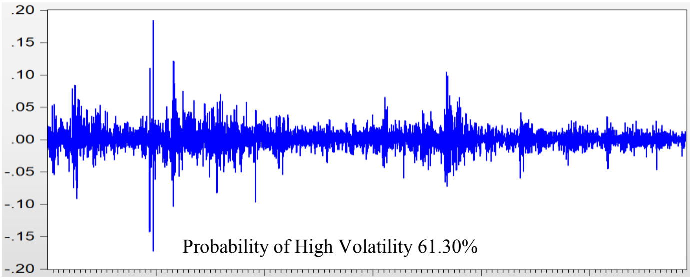
Source: Prepared by authors with Bloomberg data and E-Views.

Figure 5. Returns of Hang Seng (1994-2017)



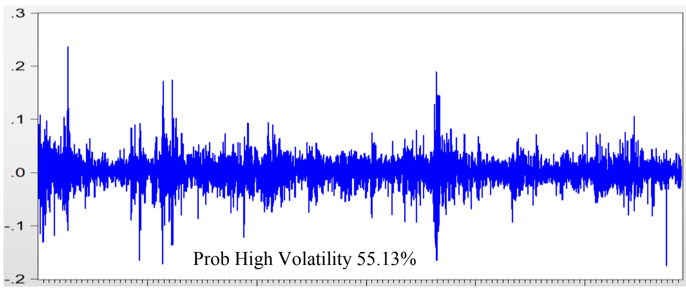
Source: Prepared by authors with Bloomberg data and E-Views.

Figure 6. Returns of IPC (1994-2017)[6]



Source: Prepared by authors with Bloomberg data and E-Views.

Figure 7. Returns of Bovespa (1994-2017)[7]



Source: Prepared by authors with Bloomberg data and *E-Views*.

The highest jump for S&P 500 is at the Subprime Mortgage Recession (Figure 1); Hang Seng shows a huge jump at Asiatic Crisis (Figure 5); For Eurostoxx50, the highest jumps are at subprime mortgage recession, Eurozone debt crisis and Brexit (Figure 2); Mexican index had the highest jump at December 1997 (Figure 6) and the highest jumps for Brazil are in the 2008 recession and in the Samba crisis (Figure 7).

6 The Mexican Index (IPC) showed an important jump at the beginning of 1995 due to a currency crisis that started in this country, known as the Tequila Effect.

7 The Brazilian Index (Bovespa) was impacted by the Samba Effect crisis at the end of 1998.

Table 2. Summary of Economic and Geopolitical Shocks (1994-2017)

Event	S&P 500	EuroStoxx50	FTSE 100	Nikkei	Hang Seng	IPC	Bovespa
Asian Financial Crisis	End of 1997 and 1998	End of 1997 and 1998	No impact	End of 1997 and 1998	End of 1997 and 1998	End of 1997	End of 1997
Dot Com Bubble	Middle of 2000 and 2001	2001 and beginning of 2002	Slightly impact	From 2000 to 2002	First semester of 2000	Middle of 2000 and 2001	End of 2001
Subprime Mortgage Recession	End of 2008 and beginning of 2009	End of 2008 and beginning of 2009	Second half of 2008	Second half of 2008	From 2007 to 2009	End of 2008 and beginning of 2009	End of 2008 and beginning of 2009
Eurozone Debt Crisis	Slightly impact in August 2008	2010 and 2011	Slightly impact	Beginning of 2011	Beginning of 2011	Slightly impact in August 2008	Slightly impact in July 2008
Brexit	Slightly impact in June 2016	June 2016	June 2016	Slightly impact in June 2016	Slightly impact on June 2016	Slightly impact in June 2016	Slightly impact in June 2016
Power takeover President Trump	Slightly impact in November 2016	Slightly impact in November 2016	Slightly impact in November 2016	2016	Slightly impact in November 2016	November 2016	November 2016

Source: Prepared by authors.

Table 2 shows the summary of economic events that have had an impact on the index returns under study.

In order to estimate the parameters of the Jump-GARCH model, the log likelihood was computed, the log likelihood of a GARCH model on the residuals was evaluated, jumps were examined, and the accumulated first and second moments were defined. Table 3 shows the estimates of the parameters of the Jump-GARCH model by using maximum likelihood for the returns of the stock indexes S&P500, EuroStoxx50, FTSE100, Nikkei, Hang Seng, IPC and Bovespa. The calibration of parameters were carried out as follows: α is the random deviation in the previous period and its average is close to 0.10; β is the variance lag and its average close to 0.90; $\alpha+\beta$ indicates the persistence, its average is more than 0.95, thus there is evidence of a GARCH effect; θ shows that jumps are related with negative movements on the price for developed economies; and γ the size of a jump; in particular Bovespa index has the highest jump intensity. The size of the jumps (%J) is between 47% and 52% higher than the previous level; S&P 500 has 52%, and Bovespa 47%. Finally, The mean (μ) of the return is the highest for IPC and the lowest for Nikkei during the period of study.

Table 3. Estimation of fractional Brownian motion combined with Jump-GARCH (Statistically significant at 95% confidence level)

Parameters	S&P 500	EuroS-toxx50	FTSE 100	IPC	Bovespa	Nikkei	Hang Seng
M	0.0002940	0.0001540	0.0004290	0.0004958	0.0002748	0.0000286	0.0001576
	0.0001365	0.0002031	0.0001737	0.0002323	0.0006977	0.0002410	0.0002626
	0.011684	0.014252	0.013178	0.015241	0.026414	0.015523	0.016204
$\%J$	52.59522	49.21371	50.42269	49.55913	47.35933	50.06818	50.05000
Λ	0.000638	0.000569	0.000498	0.000664	0.000982	0.000365	0.000533
	-0.037814	-0.013797	-0.036261	0.087828	0.065490	-0.109516	0.042823
γ	0.000001	0.000002	0.000003	0.000001	0.000016	0.000005	0.000002
	0.094715	0.081803	0.117276	0.081836	0.117298	0.101665	0.071503
	0.894829	0.911664	0.864243	0.917049	0.860306	0.880997	0.921362

Source: Prepared by authors with Bloomberg data with RATS.

The outcome of the Hurst coefficient H for the fractional Brownian motion is B_t^H is $H >$ ½ for all stock indexes, it means that the increments are positively correlateg memory (fractal markets). The estimated Hurst coefficients for all indexes are shown in Table 4.

Table 4. Hurst Coefficient

Hurst Coefficient	S&P 500	EuroStoxx50	FTSE100	IPC	Bovespa	Nikkei	Hang Seng
H	0.556061	0.533003	0.542380	0.516423	0.560925	0.539031	0.535857

Source: Prepared by authors with *R* programming.

Conclusions

For many decades, researchers have worked on describing the financial market behaviour since this is the thermometer of the economy. Relevant economics and political events (crisis) have occurred since 1929. Nowadays, the evolution of the stock market has posed new challenges such as the understanding of the behaviour of volatility of volatility and volatility clusters, thus it is required to apply new models that include sophisticated tools such as fractional Brownian motion modulated by Markov chains, in order to explain the market behaviour.

Until now, several researchers have worked on finding better models to explain the behaviour of stock returns. A substantial proportion of the variation of stock returns remains unexplained; this lack of knowledge generates uncertainty and instability, and therefore affects, not only financial markets but the economy as a whole. This paper seeks to contribute to reduce this gap by constructing a model that explains the behaviour of stock indexes volatility based on the Durham and Park's (2012) approach. The proposed extension describes the stochastic dynamics of the stock indexes of several world's main economies (US, Eurozone, UK and Japan) and some of the main emerging markets (China, Brazil, and Mexico) during 1994-2017. The outcome supports the hypothesis of long-term memory of all stock indexes, which means that the increments are positively correlated, and the series have long-term memory.

After calibrating the extension, it can be noticed that the stock indexes that have a probability over 60% to remain in high volatility are S&P 500 with 70%, and Hang Seng with 62%; while, IPC, Bovespa and FTSE100 have a high probability to stay in low volatility, 58%, 56% and 54%, respectively. The percentage of changes from high to low volatility from one period to another is just 29% for S&P 500. Nikkei has the greatest probability to move from high to low volatility but it will not remain in low volatility for long time. S&P500 and Hang Seng were found to be more volatile than other indexes. Moreover, from the GARCH estimation is observed that Bovespa and FTSE100 have the highest lag random deviation (0.1172); Hang Seng has the highest lag variance, 0.9213; Bovespa has the largest jump in size and intensity; and S&P has the greatest amount of jumps in the period studied.

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