Estocástica
FINANZAS Y RIESGO

Sovereign Bond's Credit Risk Immunization in a Tax Income Volatility Environment: The Case of a USD Denominated Mexican Bond

Salvador Cruz Aké*

Francisco Venegas-Martínez**

Agustín Ignacio Cabrera-Llanos***

Fecha de recepción: 13 de mayo de 2012
Fecha de aceptación: 16 de julio de 2012

* Instituto Politécnico Nacional, Escuela Superior de Economía. salvador.ake22@gmail.com

** Instituto Politécnico Nacional, Escuela Superior de Economía, fvenegas1111@yahoo.com.mx

*** Instituto Politécnico Nacional, UPIBI, aicllbuda@yahoo.com

Inmunización del riesgo de crédito de un bono soberano en un ambiente de ingreso fiscal volátil: el caso de un bono mexicano denominado en dólares de Estados Unidos.

Resumen

En este trabajo se utiliza el modelo de Merton (1976) de valuación de opciones y el modelo de volatilidad estocástica Heston-Nandi (2000) cuando el activo subyacente sigue un proceso de difusión con saltos para calcular las probabilidades mensuales de incumplimiento de un bono cuyo emisor tiene ingresos inciertos con alta volatilidad en la recaudación de impuestos. En particular se ilustra el caso de un bono soberano emitido por el gobierno mexicano en dólares americanos (para asegurar la existencia de riesgo de incumplimiento). La metodología propuesta incorpora los conceptos de: apalancamiento previo, capacidad de generación de ingresos, gastos no recurrentes, plazo y tamaño del préstamo (tradicionalmente usados en el cálculo de probabilidades de incumplimiento), lo que provee una metodología alternativa para el cálculo *a priori* de probabilidades de incumplimiento.

Clasificación JEL: D81, G32, F34 y G13.

Palabras clave: Probabilidades de incumplimiento, inmunización crediticia, swaps de incumplimiento de crédito.

Abstract

In this paper we use Merton's (1976) jump diffusion model and Heston-Nandi stochastic volatility model (2000) for pricing options when the underlying asset is driven by a mixed diffusion-jump process or GARCH volatility process to compute the monthly default probabilities of a bond issuer whose income is uncertain with high volatility in tax collection. In particular, we analize the case of a sovereign bond issued by the Mexican government in United States Dollars (to ensure the existence of default risk). The proposed methodology is based on concepts such as: previous leverage, income generation, non-recurring expenses, term and loan size (traditionally used in the calculation of probabilities of default), which provides an alternative methodology for computing *a priori* default probabilities.

JEL Classification: D81, G32, F34 y G13

***Keywords**: Default probability, credit risk immunization, credit default swap.*

Introduction

Endless governmental financial needs make authorities seek for funds through debt and taxation; this being their major current concern. In order to obtain the required funding, most federal governments issue debt instruments. These instruments may be backed with some specific assets such as natural resources or government owned companies, but in some cases, they have no collateral except for the fact of being issued as sovereign debt.

As any other debtor, governments are subject to default, even if bonds are issued in its own currency, despite governmental monopoly on primary money emission due to central bank constraints[1] to maintain an inflation target. This default risk is bigger when bonds are issued in a foreign currency, since the debtor cannot make use of seigniorage to fulfill their contractual liabilities. Because of this restriction, there are only two ways in which debtors are able to pay their debt, first by rolling it over (paying a loan with another one) depending on current credit conditions; second, by using their income to pay the loan.

Regardless of whether we are dealing with a company or a government, debt payment is conditioned on debtor's capacity to generate enough resources to meet his financial commitments. Fulfilling them guarantees countries and companies an easy and low cost access to debt markets, reducing thus debtor's incentives to default, so nonpayment becomes an unusual occurrence.

Although rare, defaults play an important role in the sovereign bond market due to their size. Defaults may spread across the financial system with negative political effects. Stulz (2010), Bernake, Lown and Friedman (1991), and Longstaff (2010) offer more details on spreading mechanisms across financial systems. Nonpayment marks the peak on most financial crises around the world; defaults also prelude major economic or legal changes on countries involved, Frankel and Schmukler (1996), Dungey, Fry, González-Hermosillo,

[1] *This is true in countries where the central bank is independent of the government.*

and Martin (2002), and Radelet, Sachs, Cooper and Bosworth (1998) offer more details. Due to the impact of default, it is crucial for countries and lenders to hedge against credit risk.

Financial markets have created instruments to hedge credit risk. An example is Credit Default Swaps (CDS), which intend to deal with credit risk in a similar manner to insurance. Even though CDS were created to cover corporate debt, these instruments may easily be used to cover sovereign debt if the financial system is deep enough (due to debt size) or if they are used in a deposit insurance framework.[2]

CDS works identically to its corporate counterpart. For simplicity, suppose that a country issues a sovereign bond rated BBB by some rating agency and it is bought by a single bank, called NonRiskyBank, which may be concerned about the country's capability to pay its debt. For hedging this exposure, a NonRiskyBank may enter in a deal with a counterpart called RiskyDeals who receives a regular payment, w, until the BBB's Bond expires or defaults. In that case, RiskyDeals is bound to pay to NonRiskyBank an amount equivalent to BBB's bond nominal value, N, and has the right to have the defaulted bond of which it may recover a percentage, R, of the original nominal value. Shimko (2004) or Löffler and Posch (2007) describe how CDS work.

There are several CDS valuation methods for the above described mechanism. Most of them can be grouped in two competing approaches: reduced form and non arbitrage methods. Reduced form models may be considered as bond survival models; these models concede an expected value to risky bonds given a default probability,[3] λ, and a recovery rate, R, to provide an expected fair market value on bonds default risk associated to CDS.

The reduced form CDS valuation method relies heavily on default rate probabilistic models, these include from pure structural models to pure reduced default rate probabilistic models. Altman (1968, 2000, 2002), Altman and Sabato (2007), or Merton (1974) show examples of pure structural models, since these link default probabilities with some key firm's specific financial measurements; a good survey on these models can be found on Moraux (2001), Kijima and Suzuki (2001), and Schönbucher and Schubert (2001), try to replicate the default probability stochastic process, with a pure reduced

[2] Similar to those used to hedge public deposits on private banks in most countries; an example is the Mexican Banks Savings Protection Institute (IPAB).

[3] This probability may be re-evaluated on each valuation period as in Credit Risk+.

model approach. References to traditional reduced models can be found in Davis and Lo (1999), or in Jarrow and Yu (2001).

Normality and martingale assumptions are crucial in most probabilistic CDS valuation models, in order to get a stochastic process for the default probability, in all cases they give an expected value for defaultable bonds on each coupon payment date and subtract it from the risk free bond price, in order to get an average value for the default event. Jarrow and Turnbull (1995), O'Kane (2001), O'Kane and Schloegl (2001), or O'Kane and Turnbull (2003) show more details on CDS valuation. These assumptions are very restrictive due to heavy tails existence and non independent (across time and/or creditors) default events. These shortcomings, associated with heavy tail existence, have been overcome with the use of non Gaussian copulas, as described in Schönbucher and Schubert (2001) and Crépey, Jeanblanc and Zargari (2009). Although, theoretically these models are very powerful, they have not yet been tested enough in extreme markets.

A non arbitrage approach is explained by Hull and White (2000), (2001), Blanco, Brennan and Marsh (2005), and Longstaff, Mithal and Neis (2005), and a review for this valuation method is found in Sadam (2005). The non arbitrage approach is based on complete markets assumption; this means that we can always create a synthetic portfolio that replicates a specific market asset. Thus, we have a unique risk free measure to value the risky asset; for a discussion on this topic see Lamberton and Lapeyre (1996) and Karatzas and Shreve (1991).

Complete market approach implies that only market based factors influence default on credit risky bonds, since those factors cannot be diversified, therefore they are fully discounted by market expectations. This statement is relaxed when the existence of different credit risk levels is introduced. These levels are clearly explained as a market answer to the adverse selection problem, Mas-Colell, Whinston and Green (1995).

Our model is based on the Hull and White CDS valuation model as a starting point (non arbitrage statement), and then a set of path dependent default probabilities is estimated based on Merton's jump pricing options model (non Gaussian structural model), using as input a GARCH forecast of debtor's future income and its volatility. Anything lying outside the 2σ confidence interval is considered a jump and it is included in the Merton's equation. The use of these models, considering debt payment as a strike price, transforms our model into a structural one, since debtor's income generating capability (future cash flow) is replicated by the model. This part of our methodology may be easily adapted to incorporate debtor's intrinsic variables, like Altman and Sabato (2007) do in the corporate case.

In order to show our method's consistency, we also calculate the default probabilities using the Heston and Nandi (2000) equation. In their model, these authors estimate "moneyness"[4] option's probability with an *ad hoc*[5] distribution function given by

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re^{\frac{K^{-i\phi} f^*(i\phi)}{i\phi}} d\phi,$$

where $f^*(i\phi)$ is the characteristic probability function associated to the underlying asset and its GARCH volatility process being above the strike price, *K*. Due to the characteristic function, the authors used the imaginary number, *i*.

This probability model may be used directly, since it incorporates the impact of the underlying asset GARCH process in the "moneyness" measurement. As it will be explained later, the resulting probability is slightly smaller than the one obtained by our method, because we incorporate the GARCH effect on each of Merton's probability estimation (using current data as in Heston-Nandi model), this methodology will be widely explained below.

The paper has been divided into four sections. The next section explains the possibility of modeling the default rate by means of the Merton's jump-diffusion model using debtor's income cash flow instead of considering debt as a derivative on debtor's assets. This is a key statement since it stresses that the expected income is the main uncertainty source, avoiding the use of the complex two equation nonlinear system as in the traditional Merton debt valuation model (1974) or any related estimation based on Merton's jump diffusion model (1976). The second section is devoted to explain briefly Hull and White's CDS valuation model and its interaction with previously explained

---

4 This means that the option price is above zero, therefore it will be exercided.

5 A complete explanation is given in the Heston and Nandi paper. We considered it an ad hoc model, since the characteristic distribution function, φ, changes with the GARCH process and is determined in each valuation.

probability models. Also, the role of the recovery rate calculation, R, as a non conditional expectation on debt rating will be discussed. In the third section, a hypothetical Mexican sovereign debt issuance is used to illustrate the proposed method. Eventually, we conclude explaining the results obtained and stating future research lines.

1. Stochastic Volatility and Jump Diffusion Income Associate Default Rates

As previously mentioned, our model provides a set of default rates based on debtor's income flow predictions, instead of those given by a non linear two equation system as in the traditional Merton's model (1974) or in the KMV default rate calculation methodology, see Dwyer and Stein (2004) for an insight. This approach is not as uncommon as it may appear, Sreedhar and Shumway (2008) proved that the KMV probabilities are statistically significant predictors for pricing credit sensitive securities when there is a low market volatility, but its predictive power does not arises from the non linear two-equation system, but from the default probability distribution.[6]

The predictive power associated to derivative's distribution is directly inherited from other models like Merton's jump diffusion (1976) or Heston-Nandi GARCH valuation model (2000), since in both methods, the "moneyness" probability is directly measured by the second integral of the derivative valuation.[7] Very similar results for default probabilities were obtained using both models.

Following the previous statement, we must emphasize that bankruptcy will be considered when debtor's income is not enough to fulfill its commitments on a certain (expiration) date, and not when debtor´s assets are smaller than his/her liabilities. At this point, our model resembles that of Credit Suisse Financial Products (1997) because it only considers a default as a credit event. According to Gordy (2000), we can establish a relationship between our model and Credit Metrics. In his work, Gordy shows that Credit Risk+ and Credit Metrics may be mapped one into the other by introducing a latent vari-

---

6 This was proven using a naive predictor that inherits the normal distribution but does not come from any non linear system. This naive predictor behaves slightly worse than its non linear counterparts.

7 In the case of Merton's jump diffusion model we use the $N(d_2)$, while in the Heston-Nandi model we use the P_2 probability described at the beginning of the paper.

able for default probabilities estimation, and then giving some cut off values for an associated credit rating.

Therefore, the combination of Bernoulli draws and Poisson jumps frameworks that gave rise to Credit+ may be considered as a jump-diffusion Stochastic Differential Equation (SDE) for debtor's cash flow, when regarded as a continuous function. This will be held as our main assumption, it also relates our work to some previous models that perform well, therefore it allows us to include their most important features into our framework. Furthermore, the same fact permits us to stress that we do not need to include a non linear equation system, because we are using income, I, rather than market prices for calculations. Therefore, we can use a jump-diffusion SDE for debtor's income, as

$$\mathrm{d}I_t = \mu \mathrm{d}t + \sigma \mathrm{d}W_t + \nu \mathrm{d}N_t, \tag{1.a}$$

Where μ is the instantaneous mean for income, σ is its volatility and ν is the expected jump size. Here, we have two sources of uncertainty, the Brownian motion, $\mathrm{d}W_t$ and the Poisson jump, $\mathrm{d}N_t$. The Stochastic Differential Formula in Equation (1.a) must be defined in an augmented probability space with an augmented filtration, $(\Omega, \quad , \{ \quad \}_{t \in [0,T]}, \quad)$. This construction is a simple Lévy flight that allows us to represent sudden and abrupt changes on debtor's income, (debtors may be governments or firms) during volatile or recession periods, for an insight on Lévy flights see Tankov (2011).

If the Heston-Nandi approach is used, a similar process is followed, since it implicitly states that the underlying asset (debtor's income) emulates an Instant Diffusion Equation, given by

$$\mathrm{d}I_t = \mu \mathrm{d}t + \sigma_t \mathrm{d}W_t, \tag{1.b}$$

while its volatility is a mean reversion process given by

$$\mathrm{d}\sigma_t^2 = a\left(b - \sigma_t^2\right)\mathrm{d}t + \gamma \sigma_t \mathrm{d}\mathrm{U}_t,$$

where a is the adjustment rate, b is the long run volatility, γ is the asset's volatility and dU_t is a Brownian motion associated to volatility. This Brownian motion may or may not be correlated to the underlying asset volatility; $\rho\left(\mathrm{d}U_t, \mathrm{d}W_t\right)$, obviously it is set on its own probability space given by $(\Omega^U, {}^U, \{\ \}_{t\in[0,T]}, {}^U)$.

We must state clearly that all probabilities resulting from this model were calculated using current data, so results are estimated from today to a certain point in the future, all of this included in the characteristic function, φ, used for each process. This may be partially demonstrated when comparing the Stochastic Differential Partial Equation generated by a jump diffusion process, hence

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 I_t^2 \frac{\partial^2 c}{\partial I_t^2} + rI_t \frac{\partial c}{\partial I_t} + \left[\lambda I_t \mathrm{E}_\nu\left[\nu\right]\right] \frac{\partial c}{\partial \sigma_t^2} +$$

$$\lambda \mathrm{E}_\nu\left[c\left(I_t\left(1+\nu\right)\right), t - c\left(I_t, t\right)\right] - rc = 0,$$

and those resulting from the Heston-Nandi process, given by

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_t^2 I_t^2 \frac{\partial^2 c}{\partial I_t^2} + \rho\gamma\sigma_t^2 \frac{\partial^2 c}{\partial\left(\sigma_t^2\right)^2} + rI_t \frac{\partial c}{\partial I_t} + \left[a\left(b - \sigma_t^2\right) - \gamma\sigma_t^2\right] \frac{\partial c}{\partial \sigma_t^2} - rc = 0,$$

The main difference results in a volatility change through time and the effect of this volatility in the option's vega first derivative, $\frac{\partial c}{\partial \sigma_t^2}$, this translates into the GARCH cluster effect. This effect may be partially overcome by our model using a different set of volatility and underlying values (drawn from the ARIMA-GARCH forecast) for each estimation along the coupon payment calendar, this results in the desired movements and correlations between volatility and income stated on Heston-Nandi stochastic differential partial equation (SDPE).

Differences between both SDPE may be eliminated with the correct jump tunning. In fact income jumps may be regarded as the average of volatility clusters given by $\frac{\partial c}{\partial \sigma_t^2}$, since everything outside the 2σ interval (95% of probability mass if the process is a pure Brownian motion) is considered a jump.

In other words, the log normal jump diffusion probability stated in Merton's Model (1976) using as imput some econometric ARMA-GARCH forecasts for each point of the coupon payment calendar is equivalent to a GARCH diffusion process with fixed (current) parameters estimated for each point on the coupon payment calendar. This explains the similarities between results on both approaches.

As postulated above, we use jump-diffusion SDE as stated in (1. a) to model debtor's income, *I*, in the same way as the underlying asset price is modeled in an option pricing frame. If income, *I,* is above the payment amount, *K,* on the expiration date, *T*, then the loan will not be defaulted.

Merton shows that this probability scenario matches the probability of an "in-the-money" option, so default probability is given by the probability complement. This is an important issue because we are only interested in the option probability associated, not in its financial interpretation because we are not modeling jump-option premium valuation, C_M. We only care about the probability of getting an income above debt's payment, *K*. To find this probability we can use Merton's jump diffusion option formula, given by

$$C_M\left(I_t, T-t\right) = \sum_{n=0}^{\infty} \frac{e^{-\lambda'(T-t)}\left(\lambda'(T-t)\right)^n}{n!} C_{BS,n}\left(I, T-t, K, r_n, v_n^2\right), \qquad (2)$$

where *n* is the number of lognormal jumps during the option lifetime, $\lambda' = \lambda(1+k)$ is the average number of jumps, *k* is the expected percentage change if the Poisson event soccurs, $r_n = r - \lambda k + \frac{n\gamma}{T-t}$ is the average foreign risk free rate[8] per unit of time, $\gamma = \ln(1+k)$ is an approximation of the change in jump size, $v_n^2 = \frac{\sigma^2 + n\left(\delta^2/(T-t)\right)}{T-t}$ is the average variance per unit

[8] We must remember that the bond is issued in a foreign currency, so all calculations are in that currency.

of time, σ^2 is the debtor's income volatility, δ^2 is the variance of the log normal distribution defined in Equation (1), and $C_{BS,n}$ is the Black- Scholes (1973) plain vanilla European option price for the n-th jump, such a price is given by

$$C_{BS} = I_t N(d_1) - Ke^{-r(T-t)} N(d_2), \tag{3}$$

with the appropriate substitutions, it follows

$$d_1 = \frac{\ln\left(\frac{I_t}{K}\right) + \left(r_n + \frac{v_n^2}{2}\right)(T-t)}{v_n\sqrt{T-t}},$$

and $d_2 = d_1 - v_n\sqrt{T-t}$.

As stated previously, default probabilities estimations link our model to the Credit Metrics and Credit$^+$, but also set some theoretical constraints that must be clearly stated. The most important constraint is the assumption that debtor´s income follows a log normal multivariate distribution with independent variables. This may not necessarily be true for a troubled debtor. Moreover, this assumption isolates debtor's income from other agents, complicating substantially the modeling of massive defaults effects that often impact the credit markets during recession periods. Due to capital markets global integration, massive default component is important, both to corporate debt valuation and to sovereign debtors. Recent events such as the European sovereign debt crisis confirm this assertion.

This deficiency in our model is partially overcome by the jumps and stochastic volatility included in the income modeling.[9] In Equation (2), option volatility changes when jumps are included in Merton's formula, but this adjustment is made only during option lifetime, *T-t*, since the model considers current income and volatility as fixed variables. Thus, for a pure Merton's Jump

[9] *As in all time series modeling, jump diffusion SDE modeling assumes that variable's stochastic process will maintain its interaction with the rest of the economy, which may not be true.*

Diffusion Probability Model, there is no difference whether default probabilities are estimated in a high or low volatility environment, but as explained above, according to the proposed methodology, it incorporates volatility cluster effects on default probabilities ("moneyness" probability complement) as an alternative and easier method than the Heston-Nandi approach.

We overcome the fixed volatility problem by including a GARCH model for debtor's income forecasts, as stated in Bollerslev (1986). The income forecast leads to a non constant volatility prediction that resembles the debtor's macroeconomic environment and partially incorporates the default correlation among debtors. It is possible to combine both procedures since they share the same theoretical basis, *i.e.* ergodic and stationary stochastic processes.

Following Brooks (2008), Cryer and Sik-Chan (2008) or Chan (2002), a GARCH model for debtor's income, I_t, can be stated as one ARMA process for the mean and another ARMA related process for variance. Hence

$$\begin{aligned} I_t &= c + \sum_{i=1}^{p} \Phi_i I_{t-i} + \sum_{i=1}^{q} \Theta_i I_{t-i} \sigma_{t-1} + \sigma_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{u} \alpha_i \sigma_{t-1}^2 + + \sum_{i=1}^{v} \beta_i z_{t-1}^2, \end{aligned} \tag{4}$$

Where the ARMA(p,q) model for the mean considers p autoregressive parameters, Φ_i, and q moving average parameters, Θ_i, while the ARMA for the variance has u moving average parameters, β_i, and v autoregressive parameters, α_i. In this case, the uncertainty source is given by a normal stationary stochastic process, $\{e_t\}_{t\geq 1}$, which is consistent with the Martingale framework where the Merton's model arose.[10] For this reason, normal maximum likelihood estimation is straightforward, and its results are consistent with assumptions made on the probability estimation method.

Although the forecast specification improves the correlation weakness discussed before, some GARCH family methods may deliver better results for some specific scenarios. For example, consider a medium or long run crisis environments, in this case an E-GARCH volatility specification captures the volatility behavior better. Other special case is the TARCH model used in

[10] *We should remember that Brownian motion is a stationary process.*

overshooting expectations scenarios with "relatively" common shocks. These GARCH approaches may be easily incorporated in the original proposed model.[11]

An important remark, since default probabilities are the core in CDS estimation, is that they must mirror macroeconomic environment and the debtor's capacity to fulfill his financial commitments when economic conditions change. This feature is the main improvement of our model compared with pure reduced or structural models. Our model captures intrinsic characteristics of debtor's income forecasts, and relates them with macroeconomic environment when income's volatility and average size jumps are included.

2. Credit Default Swap Calculations

The use of a CDS for a sovereign bond immunization is essentially an answer to the monitoring problem on this kind of bond, since there isn't a mechanism by which a sovereign nation may be forced to maintain certain fiscal policies in order to ensure debt payment. Actually, creditor's monitoring turns into an information mechanism that may stop credit flows into a nation, but such mechanism could be as effective as CDS spreads, due to government's commitment to be considered able to honor its debts in order to obtain new funding.

The use of the CDS spreads makes monitoring cheaper for small credit risk products investors, also allows them to diversify their portfolios with instruments that are not usually available to them. Other CDS advantage regarding sovereign debt is the possibility of risk transference among a large set of investors, through standardized contracts in big enough markets.

A market, as described in this paper, may avoid traditional problems on corporate CDS such as incentives to tear down the value associated to CDS,[12] since there are not sovereign nation shares to short selling. The only problem that investors may face is the systematic risk associated with the debt size. This case will be similar to buying private debt, since corporate bonds may be bought without holding any firm shares.

---

11 We consider that this methodological approach is not appropriate for Mexico's case because of the relative macroeconomic stability in the last decade. TARCH approach may be suitable for a stopping crisis scenario.

12 A person can lend some money to a company, buy a CDS for its debt and short their shares while they forget monitoring the company. If it defaults, the loan is hedged and they may have a profit from short selling.

Once explained the default probabilities estimations and its implications on the overall model, the CDS calculation method may be explained. Due to its simplicity and widely extended use in financial transaction, the Hull-White (2000) CDS valuation methodology, which is essentially a non arbitrage model, was chosen.

The proposed methodology was chosen over the Swap Market Model developed by Jamshidian (2004) or Libor Market Model developed by Brace, Gatarek and Musiela (1997) because of its simplicity and the possibility of incorporating jumps to the risk neutral measure associated, with each interest rate along the CDS curve obtained using a Black Scholes valuation instead of a jump diffusion model or a diffusion GARCH process. We also leave aside the Schonbucher (2000) CDS valuation model because it does not offer a risk neutral valuation default; this feature makes this methodology incompatible with the classical derivative's valuation framework.

As the reader may notice below, our model partially resembles the one proposed by Brigo (2005) where he used an option pricing equation based on a Cox process to establish an equivalence between forward rates and defaultable rates in order to value CDS with a simpler methodology.

Hull and White CDS valuation model estimates the fair payment that generates equilibrium between the present value of a risky bond expected loss (the amount received by bond holder in case of default) and a set of payments made to the insurer (the short position on CDS). This mean that CDS value is given by CDS expected payment, *CDSEP*, minus the present value of payments made to the insurer, *PVP*, hence

$$CDS = CDSEP - PVP. \tag{5}$$

Expanding Equation (5), an explicit form for Hull-White CDS valuation equation is found, hence

$$CDS = \sum_{i=1}^{n}\left(1 - R - g_{ti}R\right)Q_i v_{ti} - w\sum_{i=1}^{n}\left(u_{ti} + e_{ti}\right)Q_i - w\pi u_T, \tag{6}$$

Where T is CDS expiration date; Q_i represents default probability on last payment, given by the sum of probabilities of each period; R is the average recovery rate given default, u_t denotes the discount factor for a risky bond from today, t=0, to default time or t if the bond is not defaulted; e_t represents discount factor for a risky bond from past coupon payment, t_{i-1}, to, t_i; v_t is the discount factor for a riskless bond from today, t=0, to any given coupon payment; w is the regular payment made from CDS long to short part; π_t denotes default probability on i-th payment, given by the Merton's formula; and g_t represents accrued interest on the risky bond at t.

It is important to notice that on initial time, CDS value must be zero since we are on an arbitrage free environment, but when credit or interest rate conditions change CDS value may also change. Actually, one of the paper objectives is to estimate the regular payment amount, w*, given[13] by

$$w^* = \frac{\sum_{i=1}^{n}\left(1 - R - g_{ti}R\right)Q_i v_{ti}}{\sum_{i=1}^{n}\left(u_{ti} + e_{ti}\right)Q_i - w\pi u_T}. \tag{7}$$

This approach assumes that there is always a portfolio that may resemble risk and expected returns of any financial instrument. This implies that complete market assumption is attained, thus there is a single risk neutral probability set that allows achieving the equilibrium condition. Lamberton and Lapeyre (1996) explain the relationship between complete markets and risk neutral probabilities. This feature is the main reason to choose this particular method; it is consistent with the derivative products valuation method used previously and it can be shown that is theoretically coherent with the solution of a Stochastic Optimal Control problem for a representative agent that may invest in a jump-diffusion driven risky asset, a contingent claim on this asset and a riskless bond. A riskless bond may be related with a risky one through risk neutral option pricing, as in our model. A broader explanation is given in Venegas (2001).

The only minor change that we are proposing for the Hull and White CDS valuation method is that the recovery rate estimation, R, which is fixed on this model, is not so in our proposed method. We propose the use of an expected rate estimated through transition probabilities, p_{ij}, and their associated recovery rates, R_i, hence

[13] Interested readers may analyze all calculations and assumptions behind this equation on Hulls and White's paper.

$$R = \mathrm{E}\left[R_i\right] = \sum_{i=1}^{n} p_{ij} R_i . \tag{8}$$

The recovery rate allows us to partially capture the change in the recovery rate when the loan credit quality deteriorates, or when the whole transition probabilities curve changes due to major economic movements.

3. A USD denominated Mexican's bond

With the previous analysis for default probabilities estimation and CDS valuation, we are able to show the proposed method performance on a hypothetical USD denominated Mexican bond issued by the Mexican government. We chose this particular case since Mexico has access to global debt markets, it has no recent default history, it faces volatility problems on tax collection.

Public information on tax collection and previously issued USD denominated debt is available. The first step to carry out our calculations is to gather tax collecting data from INEGI`s[14] BIE[15] database. Tax income data (1986:01 to 2011:09) was expressed on current Mexican pesos, so monthly differences were estimated and deflated to obtain 2011 (constant) Mexican pesos. Even after these adjustments, a strong seasonal component, a clear trend and an intercept, were noticed in data, as shown in Graph 1.

After performing a KPSS stationary test, we conclude that the first seasonal differences for tax income were stationary, the resulting correlogram was examined for that stationary time series and it was concluded that the process could be modeled as driven by an ARMA (1,2,10;1,2) process. Results are shown in Table 1.

This estimation may not be parsimonious but reflects some GARCH components that clearly appear in the ARCH effects test. When those conditional volatility components are incorporated into the model, they correct the second order dependence effect and provide well fitted results.

[14] Instituto Nacional de Estadística Geografía e Informática, US Census Bureau Mexican's counterpart.

[15] Banco de Información Económica. http://www.inegi.org.mx/sistemas/bie/

Graph 1. Mexican Federal Government monthly income (1986:01 to 2011:09) in 2011 constant pesos.

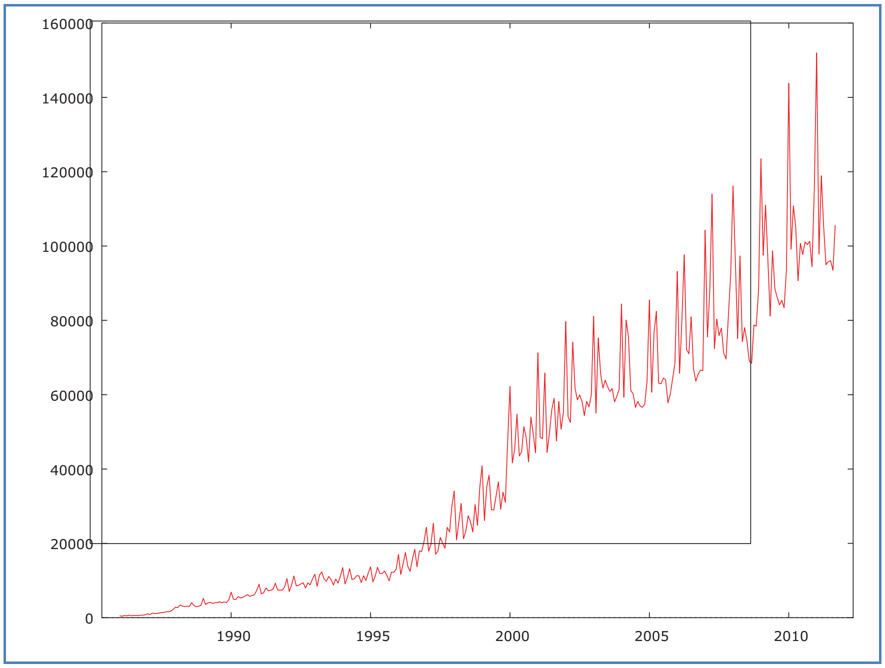


Table 1. Regression estimates for the best ARMA model.

ARMA, using 1987:01-2011:09 observations (T = 297)
Dependent variable: sd_Ingresos_Tri
Hessian based standard deviations

	Coefficient	*Standard Deviation*	*Z*	*P Value*	
Const	4078,16	1210,9	3,3679	0,00076	***
phi_1	0,268489	0,0524338	5,1205	<0,00001	***
phi_2	0,548326	0,0840448	6,5242	<0,00001	***
phi_10	0,114372	0,0523673	2,1840	0,02896	**
phi_1	0,260583	0,118093	2,2066	0,02734	**
theta_2	-0,376471	0,0981241	-3,8367	0,00012	***
theta_1	-0,635104	0,0954546	-6,6535	<0,00001	***

Dep. Variable mean.	4255,638	Deppendent D.T.	6007,021
Innovation mean	51,44147	Innovations D.T.	4926,207
Log-likelihood	-2947,998	Akaike Criterion	5911,997
Schwarz Criteron	5941,547	Hannan-Quinn Criterion	5923,827
Residuals normality test			
Null Hypothesis: Normally distributed error			
Test Statistic: Squared Chi (2) = 170,545			
P value = 9,26234e-038			
ARCH test, order 12 -			
Null Hypothesis: there is no GARCH effect			
Test Statistic: LM = 78,3302			
P Value = P(Squared Chi (12) > 78,3302) = 8,58422e-012			

The ARMA–GARCH model used to get Merton's default probabilities is similar to that established in Equation (4). Parameter estimation is showed in Table 2, and actual and fitted data are showed in Graph 2. After the model adjustment, a standard residuals test was performed finding that standard residuals and squared standard residuals are not correlated.[16]

Before estimating default probability, we must emphasize that this ARMA-GARCH model has a non explosive variance, (its coefficients are non negative and their sum is smaller than the unit), they also preserve the fitting properties showed by the previous ARMA model as shown in Graph 2.

With this model estimation at hand, debtor's (Mexican government) income in MXpesos is converted to USDollars, in order to avoid the use of seigniorage[17] to fulfill its commitment. By doing this, we assume an exchange rate risk, and eventhough this risk is not the core of our paper, it must be mentioned that this type of risk may be easily hedged with options or futures on USD/MXP.

---

[16] The residuals tests for this model are not shown but are available on request.

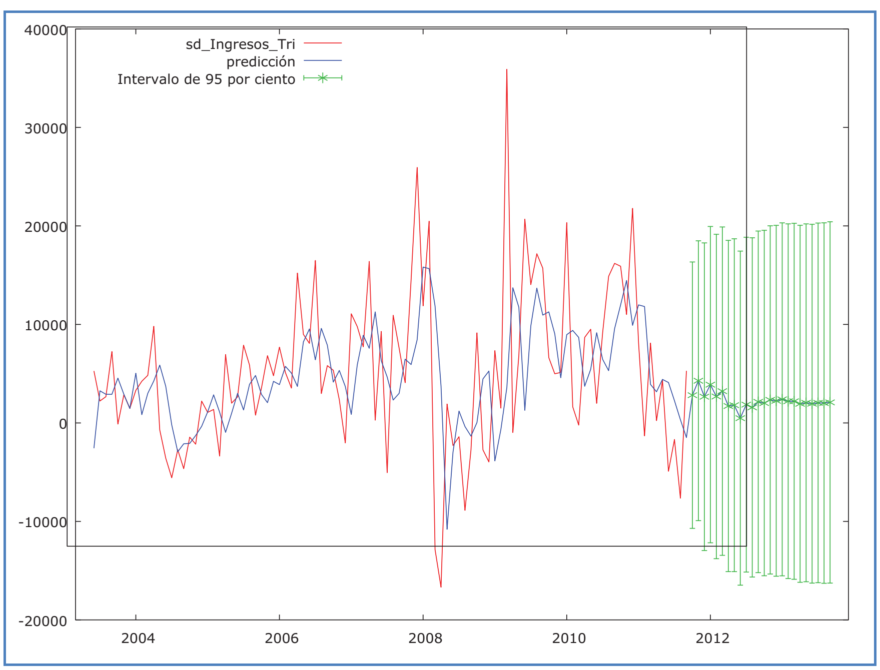
[17] This is the difference between the cost of producing Money and its value on economy.

Table 2. GARCH model parameter estimation for Mexican Tax Income.
ARMA-GARCH, using 1987:01-2011:09 observations (T = 297)
Dependent variable: sd_Ingresos_Tri
Hessian based standard deviations

	Coefficient	*Standard deviation*	*z*	*P Value*	
Const	332.545	109.509	3.0367	0.00239	***
sd_Ingresos_1	0.368048	0.0653258	5.6340	<0.00001	***
sd_Ingresos_2	0.333049	0.0781017	4.2643	0.00002	***
sd_Ingreso_10	0.141471	0.0541922	2.6105	0.00904	***
alpha(1)	0.160258	0.0616832	2.5981	0.00937	***
alpha(2)	0.170527	0.0809477	2.1066	0.03515	**

Dep. variable Mean	4379.363	Dep. variable D.T.	6073.599
Log-likelihood	-2711.740	Akaike Criterion	5443.480
Schwarz Criterion	5480.075	Hannan-Quinn Criterion	5458.147

Graph 2. GARCH Model actual vs fitted time series
for Mexican Tax Income.



All forthcoming calculations were made by using R Statistical Software.[18] The next step is to exchange income forecast into USD at a 13.4217 MXP/USD exchange rate, which represented the official fix exchange rate[19] at September 30th, 2010 (valuation date). Additionally, a linear interpolation for the implied interest rate on time t was estimated with zero-coupon risk free rate structure for both the United States and Mexico.[20]

A hypothetical monthly compounded fixed rate coupon bond issued for infrastructure development was used to show the adjustment capability of the proposed method in a realistic environment. Due to the nature of the bond, historic infrastructure expenses during the last four years were considered,[21] assuming that they will remain the same for the next 2 years[22] we estimated last 4 years average (422 millions of Mexican pesos) and doubled it for the remaining 2 years giving an 3,376,000 million investment that will be exercised monthly for the next 2 years.

Therefore, a 140,666.6 million monthly payment in Mexican pesos is considered as a 24 period annuity of 10,480, 696,395.09 USD. With this information and an interest rate linear interpolation for USD risk free interest rates, we can calculate the default rate for each monthly payment, $N(-d_2)$ with Equation (3). Using the forecasted income, I_{t+i}, as underlying asset, the monthly payment as the exercise price for a one year option,[23] the interpolated risk free rate is represented by r, the forecasted volatility σ, and the outliers number and average sizes of k and λ, hence

$$\begin{aligned} N_i\left(-d_2\right) &= f\left(S, K, T, r, b, \sigma, k, \lambda, m\right), \\ &= f\left(I_{t+i}, 10,480,696,395, 1, r_{t+i}, 0, \sigma_{t+i}, 2, .202032, 50\right). \end{aligned}$$

---

18 Freely available in http://www.r-project.org/

19 Published by Mexico's Central Bank on its web page.

20 This information was downloaded from their respective central banks.

21 Information retrieved from federal government web site http://www.infraestructura.gob.mx/index5503.html?page=requerimientos-de-inversion.

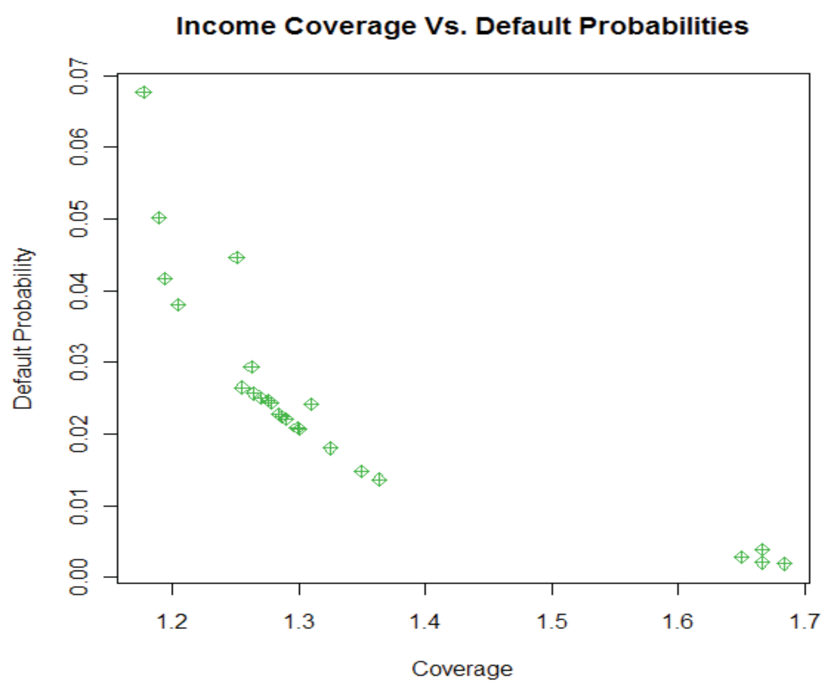
22 Assuming two years as the remaining time for current Federal Government Administration.

23 This is for getting an annual default rate.

As we showed in Equation (3) a B-S option estimation is required, so we used a slightly modified version of Option R package[24] that displays the European call or put values.

This default probability estimation method incorporates the income/debt coverage ratio, as a traditional structural model variable. Also, the proposed method incorporates income forecasted volatility and interpolated credit risk free interest rate, offering a partial view on income environment. These adjustments create the set of probability defaults showed on Graph 3.

Graph 3. Forecasted default probabilities for hypothetical Mexican bond using jump diffusion – ARIMA-GARCH probabilities method.

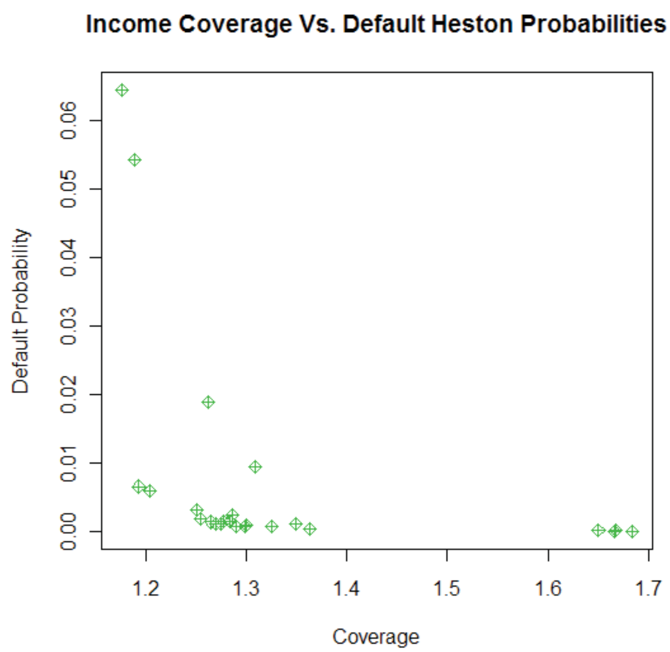


Corresponding probabilities were estimated using the Heston-Nandi framework in order to show their resemblance to the curve of jump diffusion probabilities using as input the ARIMA-GARCH forecasts, calculated with the proposed method. The Heston-Nandi estimations were made using a modified

[24] Diethelm Wuertz and many others, the package can be downloaded from http://cran.r-project.org/web/packages/Options/index.html.

Heston Nandi option original function included in the Options package in R.[25] Results are shown in Graph 4.

Graph 4. Forecasted default probabilities for hypothetical Mexican bond using jump Heston-Nandi probabilities method.



Once the jump diffusion – ARIMA-GARCH probabilities were estimated, the next step is to use Hulls & White CDS valuation formula converting the annuity present value into an equivalent coupon bond. The bond nominal value was estimated considering the current coupon rate of a previously issued Mexican bond as our coupon rate[26] (5%) and current BBB's yield[27] (6.1%) as the discount rate.

A remark on the annuity associated interest rate must be made, since it is usually calculated using a single rate for all the annuity lifetime. In our model,

25 We used a simple modification (the result is the P2 integral instead of the option value) of the HNGOption function developed by Diethelm Wuertz for the Rmetrics R-port.

26 This rate was taken from a USD denominated previous issued bond.

27 Taken from S&P webpage.

we used an average rate derived from the interpolated rates until bond´s time to maturity. Obviously, this calculation can be refined using a cubic spline interpolation for interest rates and with them estimate the Net Present Value (NPV) for each payment. This procedure is hardly used since results are about the same but the estimation strategy is far more complex.

Also, an expected value for the recovery rate as in Equation (7) is required. It was estimated based on Zazzarelli, Cantor, Emery and Truglia (2007). In their paper, they made an extensive recovery rate analysis for Moody's Investor Service that is offered at no cost on the web. Based on their estimations, a recovery rate of 39% was obtained. This rate represents the average recovery rate for a BBB debtor taking into account the possibility of change on the associated recovery rates.

With all these elements at hand, eventually CDS valuation on this hypothetical bond can be performed, resulting in a fair payment amount of 982,061,812 USD[28] for the bond with an estimated nominal value of 202,193,682,651 USD when using jump diffusion probability. If we use the Heston Nandi probability model the payment amount is 178,444,019 USD. It may seem to be a huge amount, but it is a monthly payment that represents only the 0.485% of the nominal value. If not defaulted, an 11% nominal value global payment for complete credit risk insurance will be incurred into, figure consistent with typical CDS valuations. This is the main reason why this hypothetical instrument must be issued in a deep money market in order to place it among several institutional investors, or within a global institution, typically a sovereign last resort lender such as the International Monetary Fund (IMF) or the World Bank (WB). In fact this CDS payment, w, can be considered as the fair contribution that a BBB rated nation must pay to a deposit insurance system.

The payment is slightly different from the normal distribution suggested payment due to the jumps existence, the payment was also modified by volatility induced by the GARCH approximation and the interest rate interpolation. These features have not been considered by traditional CDS valuation methods.

Conclusions

In this paper we have used the jump-diffusion risk neutral default probabilities models in a CDS valuation, we did this by using the complement of an "in-the-money" Merton's option pricing probability model where the debtor's

[28] *The algorithm used for calculations are available on request.*

income is taken as the underlying asset and the expected payment is considered as the strike price. We also incorporate debtor's income volatility into the default probabilities by using ARMA-GARCH income forecasted volatility as the underlying asset standard deviation in a jump diffusion option valuation framework. All modifications to traditional default probabilities models, usually fixed by rating agencies, as in the Hull and White CDS valuation method, result in a hybrid default probability and CDS valuation algorithm that incorporates some structural variables, as expected payment coverage, following a jump-diffusion underlying process.

Along the paper we showed that our simple jump diffusion probability series provided with ARIMA-GARCH forecasted volatilities are similar (but larger for small coverage ratios) to those obtained with the Heston-Nandi model since they arose from similar stochastic differential partial equations and the volatility process is the same, nevertheless it was calculated in different ways; outside for the jump diffusion probabilities and in the model for the Heston-Nandi ones.

The proposed method is consistent with traditional credit default probability estimation methods like Credit Metrics or Credit$^+$, and with the risk neutral valuation used in H&W method. Despite all of these advantages, the log normal jump modeling of outliers may be improved, and perhaps it can be modeled by extreme value jump-diffusion processes. This is left as a future research as well as the use of these default probabilities in a reduced CDS valuation model or in a copula based probability model.

Bibliography

Altman, E. I. (1968). "Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy". *Journal of Finance*. Vol. 23, No. 4, pp 189–209.

______________. (2000). "Predicting Financial Distress of Companies". http://pages.stern.nyu.edu/~ealtman/Zscores.pdf. pp. 15–22.

______________. (2002). "Revisiting Credit Scoring Models in a Basel II Environment". Credit Rating: *Methodologies, Rationale, and Default Risk*, London Risk Books.

______________. and G. Sabato (2007). "Modelling Credit Risk for SMEs: Evidence from the U.S. Market". *Abacus*, Vol. 43, No. 3, pp 332-357

Bernanke, B.S., C. S. Lown and B. M. Friedman. (1991). "The credit Crunch". *Brookings Papers on Economic Activity*, No. 2, pp. 205-247.

Black, F. and M. Scholes. (1973). "The Pricing of Options and Corporate Liabilities". *Journal of Political Economy*, Vol. 81, No. 3, pp 637–654.

Blanco R., S. Brennan, and I. W. Marsh (2005). "An Empirical Analysis of the Dynamic Relation between Investment-Grade Bonds and Credit Default Swaps". *The Journal of Finance*, Vol. 60, No. 5, pp. 2255–2281

Bollerslev, T. (1986). "Generalized Autoregressive Conditional Heteroskedasticity", *Journal of Econometrics*, Vol. 31, No. 3, pp. 307-327

Brace, A., Gatarek, D. et Musiela, M. (1997). "The Market Model of Interest Rate Dynamics", *Mathematical Finance*, 7(2), pp. 127-154.

Brigo, D. (2005). "Market Models for CDS Options and Callable Floaters", *Risk, January*.

Brooks, C. (2008). "Introductory Econometrics for Finance", 2nd Edition. *Cambridge University Press*.

Crépey, S., M. Jeanblanc, B. Zargari (2009). "CDS with Counterparty Risk in a Markov Chain Copula Model with Joint Defaults". Département de Mathématiques Université d'Évry Val d'Essonne. *Working papers*.

Chan, N. H. (2002)." Time Series", *Applications to Finance*. Wiley Series in Probability and Statistics. USA.

Credit Suisse Financial Products (1997). CreditRisk+: A Credit Risk Management Framework, London. Credit Suisse Financial Products.-..T,

Cryer, J.D. and K. Chan. (2008). "Time Series Analysis with applications in R. 2nd Edition Springer". USA.

Davis, M., and Lo, V., (2001). "Infectious Defaults," *Quantitative Finance*, Vol.1, pp. 382-386.

Dungey, M., F, R., González Hermosillo, B., Martin, V. (2002). "International Contagion Effect from the Russian Crisis and the LTCM Near Collapse". *IMF Working Papers*.

Dwyer, D. and R. Stein (2004). Technical Document on RiskCalc v3.1 Methodology. Moody's – KMV.

Frankel, J.A. and S.L. Schmukler (1996). "Country fund discounts and the mexican crisis of December 1994: Did local residents turn pessimistic before international investors?" *Open Economies Review*, Vol. 7, Supplement 1, pp. 511-534.

Gordy, M. (2000). "A comparative anatomy of credit risk models". *Journal of Banking & Finance*, Vol. 24, No.1-2, pp. 119-149.

Heston, S. and S. Nandi (2000). "A Closed Form GARCH option Valuation Model". *The review of Financial Studies*. Vol. 13, No. 3. pp 327 – 343.

Hull, J. C. and A. White (2000). "Valuing Credit Default Swaps I: No Counterparty Default Risk". *Journal of Derivatives*, Vol. 8, No. 1, pp. 29-40

__________. "Valuing Credit Default Swaps II: Modeling Default Correlations". *Journal of Derivatives*, Vol. 8, No. 3. pp. 12-22.

Karatzas, I. and S. Shreve (1991). "Brownian Motion and Stochastic Calculus". 2nd Ed. Springer.

Kijima, M. and T. Suzuki (2001). "A jump diffusion model for pricing corporate debt securities in a complex capital structure", *Quantitative Finance*, Vol. 1, No. 6, pp. 611-620.

Jarrow, R., and Yu, F., (2001). "Counterparty Risk and the Pricing of Defaultable Securities," *Journal of Finance*. Vol. 56, pp 1765-1799.

__________. and Turnbull (1995). "Pricing Derivatives on Financial Securities Subject to Credit

Risk", *Journal of Finance*, Vol. 50 (1995), 53-85.

Jamshidian, F. (2004). "Valuation of Credit Default Swaps and Swaptions". *Finance*

and Stochastics. Vol 8, pp 343-371.

Lamberton, D. and B. Lapeyre (1996). "Introduction to Stochastic Calculus Applied to Finance". Champan & Hall.

Longstaff, F. A., S. Mithal, and E. Neis (2005). "Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Default Swap Market". *The Journal of Finance*, Vol. 60, No. 5, pp. 2213–2253.

__________ (2010). "The subprime credit crisis and contagion in financial markets". *Journal of Financial Economics*, Vol. 97, No. 3. pp. 436-450.

Löffler, G. and P. N. Posch (2007). "Credit risk modeling using Excel and VBA". *John Wiley and Sons*. USA.

Mas-Colel, A., M. Whinston and J. Green (1995). "Microeconomic Theory". *Oxford University Press*.

Merton, R. C. (1974). "On the pricing of corporate debt: The risk structure of interest rates". *Journal of Finance*. Vol. 29, No. 2, pp. 449-470.

__________. (1976). "Option Pricing when underlying stock returns are discontinuous", *Journal of Financial Economics*. Vol. 3, pp 125 – 144.

Moraux, F. (2001). "Structural Credit Risk Models for Defaultable Securities". *Working Paper*, Université de Rennes I-IGR.

O'Kane (2001). Credit Derivatives Explained, Lehman Brothers, March 2001.

______________, and Schloegl (2001). "Modelling Credit: Theory and Practice", February 2001.

O'Kane and Turnbull (2003). "Valuation and Risk-Management of Credit Default Swaps", *Lehman Brothers*

Radelet, S., J. D. Sachs, R.N. Cooper and B.P. Bosworth (1998). "The East Asian Financial Crisis: Diagnosis, Remedies, Prospects". *Brookings Papers on Economic Activity.*

Shimko, D. (2004). "Credit Risk Modeling and Valuation: An Introduction in Credit Risk". *Models and Management*. Vol. 2, pp. 487–525,

Sadam, I. (2005). Alkalmazott matematikus szak, Thesis, Természettudományi Kar.

Sreedhar T. B. and T. Shumway (2008). "Forecasting default with the Merton distance to default model". *Review of Financial Studies*, Vol. 21, No. 3. pp. 1339-1369.

Schönbucher, P. (2000). "A Libor market model with default risk", *preprint.*

______________. and D. Schubert (2001). "Copula-Dependent Defaults in Intensity Models", Available at SSRN: http://dx.doi.org/10.2139/ssrn.301968

Stulz, R. M. (2010). "Credit Default Swaps and the Credit Crisis". *Journal of Economic Perspective*, Vol. 24, No. 1. pp. 73–92.

Tankov, P. (2011). "Financial modeling with Levy processes". *Centre de Mathèmatiques Appliquèes Ecole Polytechnique*. Lecture Notes.

Venegas-Martinez, F. (2001). "Temporary Stabilization: A stochastic Analysis". *Journal of Economic Dynamics and Control*, Vol. 25, No. 9, pp 1429-1449.

Wuertz, D., *et al* (2012). "Options Package for the Rmetrics R-port". GPL licence.

Zazzarelli, A., R. Cantor, K. Emery and V. Truglia (2007). "Sovereign Default and Recovery Rates1983 - 2006". Moody's I.