Generating covariances in multifactor CIR model

Wojciech Szatzschneider*

Fecha de recepción: 12 de abril de 2013
Fecha de aceptación: 12 de diciembre de 2013

* Universidad Anáhuac, México Norte
Escuela de Actuaría
wojciech@anahuac.mx

Generación de covarianzas con el modelo multifactorial CIR

Resumen

Se presenta el marco general para generar covarianzas entre instrumentos con tasas de interés libre de riesgo r(t) e instrumentos con intensidad de incumplimiento $\lambda(t)$, en el modelo Cox, Ingersoll, Ross (CIR) o en el modelo extendido CIR multifactorial.

Clasificación JEL: C15, C58, C63

Palabras claves: Modelo CIR, modelo multifactorial -para tasa de interés, Teorema de Girsanov.

Abstract

This paper presents a general framework of how to generate covariances between riskless interest rate $r(t)$ instruments, and financial instruments with intensity of default $\lambda(t)$, in Cox, Ingersoll, Ross (CIR), or in the extended multifactor CIR model.

JEL classification: C15, C58, C63

Keywords: *CIR model, Multifactor model for interest rate, Girsanov theorem*

Introduction

The problem of generating covariances between intensity of default and riskless interest rates that are consistent with observed data has been treated in several theoretical and practical settings, [Duffie, 2011]. For definitions and resulting formulas refer to the second printing of [Bielecki & Rutkowski, 2004]. In the quoted book the general construction of extended Cox, Ingersoll, Ross (CIR) can be found [Szatzschneider, 2002]. In section 1 this method is briefly sketched. Although the use of covariances very often leads to poor measurement of real dependencies, it is assumed that only dependences observed are covariances. A free dependence structure between $r(t)$ and $\lambda(t)$ is proposed to obtain explicit, or almost explicit results for prices of riskless bonds, defaultable bonds and covariances.

Sections 1 and 2 present a general method for generating covariances for predefined functions, particularly polynomial ones. Section 2 explains how to deal with plausibly observed negative covariances. Sections 3 and 4 analyze the problem for some classes of functions. It is made clear what can or cannot be done in a CIR setting.

Several relevant quotations shall be considered:

1. $r(s)$ and $\lambda(s)$ have negative (observed) correlation -20% [Schönbucher, 2003].
2. CIR (CSR) correlated square root models are theoretically incapable of generating negative correlations [Dai & Singleton, 2000].
3. The dynamics of r and λ are rich enough to allow for a realistic description of the real-world prices [Schönbucher, 2003].

The usual construction of the multifactor model for interest rate and intensity of default is as follows:

$$r = m_1 X_1 + m_2 X_2 + \ldots + m_n X_n$$

$\lambda = \overline{m}_1 X_1 + \overline{m}_2 X_2 + \ldots + \overline{m}_n X_n$, X_i independent (positive) CIR models and scalars *m*´s are nonnegative. This setting is called “usual construction”.

1. Model constraints

1. It is not possible to obtain general covariance structure (even positive) if X_i CIR. Using comparisons theorems for diffusions, one can prove easily that

$$0 \le Cov(r(t), \lambda(t)) \le At^2 + Bt + C .$$

2. The second constraint is the following.

   t^5 cannot be uniformly approximated in $[0,1]$ by $Cov(r(t), \lambda(t))$. Using well known formulas such as:

$$Cov(r(t), \lambda(t)) = \sum_i m_i \overline{m}_i Var(X_i) = \sum_{i=1}^{n} A_i \left(e^{-k_i t} - e^{-2k_i t}\right) + B_i \left(1 - e^{-k_i t}\right)^2 ,$$

$$k_i, A_i, B_i > 0 .$$

   An elementary but somewhat tricky proof follows:

$$\int_0^1 e^{-kt}\left(1 - e^{-kt}\right) dt \ge \frac{1}{2} e^{-kt}\left(1 - e^{-kt}\right)$$

   because:

$$\frac{e^{-kt}}{e^{-k}} \left(\frac{1 - e^{-kt}}{1 - e^{-k}} \right) \ge t$$ resulting from:

$$\frac{e^{-kt}}{e^{-k}} \geq 1 \text{ for } t \leq 1 \text{ and,}$$

$$\frac{1-\left(e^{-k}\right)^t}{1-e^{-k}} = tc^{t-1} \geq t \text{ for some } c \in \left(e^{-k},1\right)$$

and

$$\left(\int_0^1 \left(1-e^{-kt}\right)^2 dt\right)^{\frac{1}{2}} \geq \int_0^1 \left(1-e^{-kt}\right) dt \geq \frac{1}{2}\left(1-e^{-k}\right).$$

therefore

$$\int_0^1 \left(1-e^{-kt}\right)^2 dt \geq \frac{1}{4}\left(1-e^{-k}\right)^2,$$

then, for $f(t)$ any linear combination

$$\int_0^1 f(t)\,dt \geq \frac{1}{4} f(1)$$

and for $g(t) = t^5$, we have

$$\int_0^1 g(t)\,dt = \frac{1}{5} g(1)$$

3. The third constraint is clear: Negative covariances cannot be generated from the usual construction , [Dai & Singleton, 2000 provide further discussion].

The easiest way to obtain negative correlations seems to be,

$$X_i \sim \text{driven by } W_i$$

$$Y_i \sim \text{driven by } -W_i\,.$$

But there is no possibility to get explicit results, the only possible approach is process simulation.

2. Particular method and extended CIR

"Explicit" formulas are desirable for:

$$B(0,t) = E\left(e^{-\int_0^t r(s)ds} \right) \tag{1}$$

$$\overline{B}(0,t) = E\left(e^{-\int_0^t (r(s)+\lambda(s))ds} \right) \tag{2}$$

$$Cov(r(t), \lambda(t)) \tag{3}$$

To be able to reproduce a given arbitrary covariance structure, an extended CIR Model (ECIR) with time dependent parameters has to be used.

First a short, user friendly construction of extended CIR with references quoted in the introduction is presented.

1 Start from $BESQ^{\delta}$.

$$dX(t) = 2\sqrt{X(t)}\, dW(t) + \delta\, dt,\ X(0)>0. \tag{4}$$

2. Add the drift $2\beta_t \cdot r(t)$ (Girsanov).

3. Multiply the process by σ_t

Now,

$$dr(t) = 2\sqrt{\sigma_t \cdot r(t)}\, dW(t) + \left\{ \left[2\beta_t + \frac{\sigma'(t)}{\sigma(t)} \right] r(t) + \delta\, \sigma_t \right\} dt$$

and

$$B(0,t) = \varphi^{\frac{1}{2}}(t)\exp\left[-\left(\int_0^t \beta_s ds\right)\delta + x\left(-\beta_0 + \varphi'(0)\right)\right]$$

$$x = \frac{r_0}{\sigma_0}$$

and

$$\frac{\varphi''(s)}{\varphi(s)} = h(s) = \beta_s^2 + \beta_s' + 2\sigma_s \text{ in } [0,t]$$

$$\frac{\varphi'(t)}{\varphi(t)} = \beta_t,\ \varphi(0) = 1 \text{ (Sturm-Liouville equation),}$$

or equivalently in terms of Riccati equation

$$F^2(s) + F(s) = h(s),\ F(t) = \beta_t,\ F(s) = \frac{\varphi'(s)}{\varphi(s)}.$$

Therefore, explicit formulas for bonds prices depend on the solutions of these equations.

3. Generating positive covariances for polynomials

Note that the free term of polynomials should be zero.

For grade 3 polynomials set F(t)=0, fix time t, and

$$r_i(s) = X_i(s)\cdot\varepsilon,\ X_i \sim BESQ^0,$$

$$\lambda_i(s) = \sigma_i(s)\cdot X_i(s),$$

$$\lambda_i(s) + r_i(s) = \left(\sigma_i(s) + \varepsilon_i\right)X_i(s).$$

For a moment only one factor will be considered.

Choose $F(s) = D(s-t)$, for s<t, D being constant.

$$F'(s) + F^2(s) = D + D^2(s-t)^2$$

$$\sigma(s) := D + D^2 s^2 - 2dst + t^2 D^2 - \varepsilon$$

and assume that $\sigma(s) > 0$.

If $X(s) \sim BESQ^{\delta}$

$$Var(X(s)) = 4sX(0) + 2\delta s^2$$

and

$$Cov(X(s), \lambda(s)) = \varepsilon \cdot \sigma(s) \cdot Var(X(s))$$

Elementary calculations show that one can generate any positive covariances as polynomial of grade 3 from $BESQ^0$ using two factors.

For grade 4 polynomials the same procedure applies but starting with $BESQ^{\delta}$ instead of $BESQ^0$, $\delta > 0$.

For grade 5 polynomials $F(s) = D(s-t)^2$, and so on.

This method leads to a general construction for any positive covariance structure, and equations (1), (2), (3) can be obtained.

4. Negative correlation

Take factors as $BESQ^1$, more explicitly:

$$r(t) = (W(t) + A)^2,$$

$$\lambda(t) = (W(t) - B)^2, \; A, B > 0.$$

Both being $BESQ^1$ process driven respectively by:

$$B_1(t) = \int_0^t \text{sgn}(W(s) + A)\, dW(s)$$

$$B_2(t) = \int_0^t \text{sgn}(W(s) - B)\, dW(s)$$

Now $Cov(\lambda(t), r(t)) = 2t(t - AB)$, is negative for $t < AB$.
$(W(t) + A)^2 + (W(t) - B)^2 = constant + 2BESQ^1$ starting at $\frac{(A - B)^2}{2} = x$, so any quadratic polynomial with explicit formulas (1),(2),(3) can be generated.

Grade 3 polynomials with some restrictions can be generated as a combination of the results of this section and the previous one.

Grade 6 polynomials can be obtained for example multiplying by $\sigma(s)$, $\sigma(s)$ as before (but without ε)

$$\sigma(s) = D + D^2 s^2 - 2Dst + t^2 D^2$$

obtained from $F(s) = D(s - t)$.

5. Other explicit constructions

Set $\beta(s) = \frac{1}{s + A}$, $A < 0, t + A > 0$

$$\beta^2(s) + \beta'(s) = 0$$

$$r(s) = X(s) \cdot \varepsilon,\ X(s) \text{ CIR with drift } \beta(s), \sigma = 1,$$

$$\lambda(s) = X(s) \cdot \sigma(s).$$

For $B(0,t)$

$$\frac{\varphi''(s)}{\varphi(s)} = 2$$

$$\frac{\varphi''(s)}{\varphi(s)} = \beta_t$$

$Cov\,(\lambda(t), r(t)) = \sigma(s)\cdot \varepsilon \cdot Var(X(t))$, σ still undefined.

Set $V_t \sim BESQ^{\delta}$, $V_0 = x$

$$E\left(X^2(t)\right) = E\left(V_t^2 \cdot e^{\frac{1}{2}\beta_t V_t}\right)\cdot \text{constant} = f(t)\cdot \int_0^{\infty} x^2 e^{\frac{1}{2}\beta_t x} f_{V(t)}(x)dx\,.$$

The density $f_{v(t)}$ is well known.

$$\overline{B}(0,t) = E\left(\exp\left(-\int_0^t (\sigma(s)+\varepsilon)X(s)\,ds\right)\right)$$

$$= E\left(\exp\left(\frac{1}{2}\beta_t X_t - \int_0^t X(s)(\sigma(s)+\varepsilon)\,ds\right)\right)$$

Set

$$F(s) = \beta_t + D(s-t),$$

and explicit results can be

$$\sigma(s) = F^2(s) + F'(s) - \varepsilon \text{ assuming } \sigma(s) > 0\,.$$

6. More general construction

$r_i(s) = \frac{1}{2}\sigma(s)Y(s) = \lambda_i(s)$ for some factor

$2\sigma(s) = c^2 - \beta^2(s) - \beta'(s) > 0$, for some choice of $\beta(s)$

$\overline{B}(0,t)$ is explicite (CIR).

However in this case $B(0,t)$ not explicit.

Calculation of covariances:

The problem is easier if based initially on $BESQ^0$.

$$dY(t) = 2\sqrt{Y(s)}\, dW(s) + 2\beta(s)Y(s)ds\,, \qquad \beta(\mathbf{s}) < 0$$

$$Y(0) = 1 \text{ for example}$$

$$E(Y(t)) = \exp\left(2\int_0^t \beta(s)ds\right)$$

$$E\left(Y^2(t)\right) = H(t), \text{ where}$$

$$H''(t) = 4H(t) + \beta(t)e^{2\int_0^t \beta(s)d}$$ and can be solved explicitly.

Other general modeling possibility uses Laplace transform for the process $Y(t)$ for general $\beta(s)$. In this case,

$$F^2(s) + F'(s) = \beta^2(s) + \beta'(s),\ \beta(s) = \frac{\varphi'(s)}{\varphi(s)}.$$

But now $F(t) = \beta(t) + \lambda$ for some λ (⋆) and general solution of this equation can be obtained solving:

$$F(s) = \frac{\psi'(s)}{\psi(s)}$$

$$\psi(s) = \varphi(s)\left(1 + A\int_0^s \frac{1}{\varphi^2(u)}du\right)$$

One can find *A* to satisfy ⋆ and get explicit formula for $E\left(e^{-\lambda Y(t)}\right)$ with $\sigma(t) = \sigma = 1$ but not for the price of the bond. This model is clearly quite difficult to put into practice. A rather complicated density of CIR is presented in [Jeanblanc *et al*, 2009] (p.358).

Conclusions

As has been shown, there are many methods to generate given (observed) covariance structure between instantaneous riskless interest rates, and intensity of default. However, to obtain user friendly results in the case of negative correlations, one should not expect substantial extensions of presented use of “degenerated” CIR´s- squares of Brownian motions.

As a final comment, the CIR model is very attractive and interesting, being the “Girsanov version “of square of Brownian Motion, but it has generated in the past many erroneous formulas. See for example the excellent textbook by [Jeanblanc *et al*, 2009] p. 127, where an erroneous application of Ito´s formula appears, this mistake is explained extensively in [Szatzschneider 2008].

References

Darrell Duffie (2011). *Measuring Corporate Default Risk*. Oxford University Press.

Jeanblanc Monique, Yor Marc, Chesney (2009). *Mathematical Methods for Financial Markets*. Springer-Verlag.

Dai Q. and Singleton K (2000). “Specification Analysis of Affine Term Structure Models”. *Journal of Finance* 55, 1943-1978.

Bielecki T. and Rutkowski M. (2004). “Credit Risk: Modeling, Valuation and Hedging”. *Springer*.

Schönbucher P. (2003). “Credit Derivatives Pricing Models: Models, Pricing and Implementation”. Wiley, p. 175.

Szatzschneider W. (2002). “The Cox, Ingersoll and Ross Extended Model”. *Revista Mexicana de Economia y Finanzas* 1, 319-332.

______________ (2008). “Exponential Martingales and CIR Model”. *Advances in Mathematics y of Finance*, Banach Center Publications Vol. 83 pp. 243-249.